Theorem (Euler–Maclaurin formula).

Let $f : [y, x] \to \mathbb{C}$ a C^1-function. Then

$$\sum_{y < n \le x} f(n) = \int_y^x f(t)\,dt + \int_y^x f'(t)\psi(t)\,dt + f(y)\psi(y) - f(x)\psi(x),$$

where $\psi(t) = \{t\} - \frac{1}{2} = t - \lfloor t \rfloor - \frac{1}{2}$.

Proof:

Claim: Let $n \in \mathbb{Z}$. Then

$$\frac{f(n) + f(n+1)}{2} = \int_n^{n+1} f(t)\,dt + \int_n^{n+1} \psi(t) f'(t)\,dt.$$

Proof of claim:

$$\int_n^{n+1} \psi(t) f'(t)\,dt = \int_n^{n+1} \left(t - n - \frac{1}{2}\right) f'(t)\,dt$$

$$= \int_n^{n+1} t f'(t)\,dt - \left(n + \frac{1}{2}\right) \int_n^{n+1} f'(t)\,dt$$

$$= \Big[t f(t)\Big]_n^{n+1} - \int_n^{n+1} f(t)\,dt - \left(n + \frac{1}{2}\right) \Big[f(t)\Big]_n^{n+1}$$

$$= (n+1) f(n+1) - n f(n) - \int_n^{n+1} f(t)\,dt - \left(n + \frac{1}{2}\right)\left(f(n+1) - f(n)\right)$$

$$= \frac{f(n+1)}{2} + \frac{f(n)}{2} - \int_n^{n+1} f(t)\,dt. \quad \checkmark$$

Case 1: y, x integers. We sum the quantities in the claim for all integers $y \le n < x$.

We get
$$\frac{f(y)}{2} + \sum_{y < n < x} f(n) + \frac{f(x)}{2} = \int_y^x f(t)\,dt + \int_y^x \psi(t) f'(t)$$

$$\Rightarrow \sum_{y < n \le x} f(n) = \int_y^x f(t)\,dt + \int_y^x \psi(t) f'(t) + \frac{f(x)}{2} - \frac{f(y)}{2}$$

claim follows from the fact that $\psi(k) = -\frac{1}{2}$, for $k \in \mathbb{Z}$.

Case 2: General y, x.

Exercise:
$$\int_y^{\lfloor y \rfloor + 1} \psi(t) f'(t)\,dt = -\psi(y) f(y) + \frac{f(\lfloor y \rfloor + 1)}{2} - \int_y^{\lfloor y \rfloor + 1} f(t)\,dt$$

$$\int_{\lfloor x \rfloor}^x \psi(t) f'(t) = \psi(x) f(x) + \frac{f(\lfloor x \rfloor)}{2} - \int_{\lfloor x \rfloor}^x f(t)\,dt$$

$\boxtimes$

Example: $\displaystyle\sum_{n \le x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$,

$\gamma = 0.57721\ldots$ Euler-Mascheroni constant

Theorem (Abel summation or Partial summation)

Let $f \in \mathcal{R}$, $0 < y < x$, $g \in C^1([y,x])$.

Define $F(T) := \sum_{n \le T} f(n)$.

Then $$\sum_{y < n \le x} f(n) g(n) = F(x)g(x) - F(y)g(y) - \int_y^x F(t) g'(t)\,dt$$

Proof: WLOG $\lfloor y \rfloor + 1 \le \lfloor x \rfloor$

(otherwise no integer between y and x, LHS $= 0$ and $F(x) = F(y) = F(t)$, $\forall t \in [y, x]$, so RHS $= 0$).

Then
$$\sum_{y < n \le x} f(n) g(n) = \sum_{\lfloor y \rfloor + 1 \le n \le \lfloor x \rfloor} (F(n) - F(n-1)) g(n)$$

$$= \sum_{\lfloor y \rfloor + 1 \le n \le \lfloor x \rfloor} F(n) g(n) - \sum_{\lfloor y \rfloor \le n \le \lfloor x \rfloor - 1} F(n) g(n+1)$$

$$= F(x) g(\lfloor x \rfloor) - F(y) g(\lfloor y \rfloor + 1) - \sum_{\lfloor y \rfloor + 1 \le n \le \lfloor x \rfloor - 1} \underbrace{F(n)(g(n+1) - g(n))}_{\left(\int_n^{n+1} F(t) g'(t)\,dt\right)}.$$

$$= F(x) g(\lfloor x \rfloor) - F(y) g(\lfloor y \rfloor + 1) - \int_{\lfloor y \rfloor + 1}^{\lfloor x \rfloor} F(t) g'(t)\,dt$$

Now we see $\int_{\lfloor x \rfloor}^{x} F(t) g'(t) dt = F(x)(g(x) - g(\lfloor x \rfloor))$

$$\int_{y}^{\lfloor y \rfloor + 1} F(t) g'(t) dt = F(y)(g(\lfloor y \rfloor + 1) - g(y)).$$

∎

<u>Corollary</u>: Let $g : [y, \infty) \to \mathbb{C}$, $f \in \mathbb{R}$ and suppose $F(x) g(x) \to 0$ as $x \to \infty$.

Then $$\sum_{n \geq y} f(n) g(n) = - \int_{y}^{\infty} F(t) g'(t) dt - F(y) g(y)$$

whenever both sides converge.

<u>Example</u>: $\exists C' > 0$ s.t. $\sum_{n \leq x} \frac{\varphi(n)}{n} = C'x + O((\log x)^2)$.

<u>Pf</u>: Denote $F(y) = \sum_{n \leq y} \varphi(n)$. From last time,

$$F(y) = C \cdot y^2 + O(y \log y).$$

Hence by partial summation,

$$\sum_{n \leq x} \frac{\varphi(n)}{n} = 1 + \frac{F(x)}{x} - 1 + \int^{x} F(t) \cdot \frac{1}{t^2} dt$$

$$= \frac{cx^2 + O(x \lg x)}{x} + \int_1^x \frac{c \cdot t^2 + O(t \lg t)}{t^2}\, dt$$

$$= c \cdot x + O(\lg x) + c(x-1) + O\left(\int_1^x \frac{\lg t}{t}\, dt\right)$$

$$= 2 \cdot c \cdot x + O(\lg x) + O\left([\log t]^2\Big|_1^x\right)$$

$$= 2 \cdot c \cdot x + O((\log x)^2).$$ ☺

__Definition__: The von Mangoldt function $\Lambda(n)$ is

$$\Lambda(n) := \begin{cases} \log p, & \text{if } n = p^l,\ p \text{ prime},\ l \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

__Lemma__: $\Lambda = \log * \mu$.

__Proof__: Need to show $\Lambda(n) = \log * \mu(n)$, $\forall n \in \mathbb{N}$

$n = 1$ ✓

Let $n = p_1^{l_1} \cdots p_k^{l_k}$, distinct primes p_i, $l_i \in \mathbb{N}$.

$$\Lambda * \varepsilon(n) = \sum_{d|n} \Lambda(d) = \sum_{i=1}^{k} \sum_{j=1}^{l_i} \Lambda(p_i^{k})$$

$$= \sum_{i=1}^{k} l_i \log(p_i) = \log(p_1^{l_1} \cdots p_k^{l_k}) = \log n.$$ ☒

Definition: $\psi(x) := \sum_{n \leq x} \Lambda(n)$

$$\vartheta(x) := \sum_{n \leq x} \mathbb{1}_p(n)\Lambda(n) = \sum_{p \leq x} \log p.$$

Lemma: $\psi(x) = \vartheta(x) + O(\sqrt{x}(\log x))$.

Proof:

$$0 \leq \psi(x) - \vartheta(x) = \sum_{\substack{p \text{ prime} \\ m \geq 2,\, p^m \leq x}} \log p$$

$$= \sum_{p \leq \sqrt{x}} \log p \left\lfloor \frac{\log x}{\log p} \right\rfloor \rightarrow p^m \leq x \Rightarrow m \leq \frac{\log x}{\log p}$$

$\hookrightarrow p^m \leq x$, for $m \geq 2 \Rightarrow p \leq \sqrt{x}$

$$\leq \sum_{p \leq \sqrt{x}} \log p \cdot \frac{\log x}{\log p} \leq \sqrt{x} \log x. \qquad \blacksquare$$

Theorem: The following are equivalent:

(i) $\pi(x) \sim \frac{x}{\log x}$ (PNT)

(ii) $\psi(x) \sim x$

(iii) $\vartheta(x) \sim x$

Proof: We've already seen (ii) $\Leftrightarrow$ (iii).

"(i) $\Rightarrow$ (iii)" Assume $\pi(x) \sim \frac{x}{\log x}$.

By Abel summation ($f = 1_P$, $g = \log$)

$$\sum_{p \le x} \log p = \pi(x) \log x - \int_1^x \frac{\pi(t)}{t}\, dt$$

$\pi(t) = 0$ for $t < 2$

$$= \left(\frac{x}{\log x} + o\left(\frac{x}{\log x}\right)\right) \log x - \int_2^x \frac{\pi(t)}{t}\, dt$$

Note $\int_2^x \frac{\pi(t)}{t}\, dt \ll \int_2^x \frac{1}{\log t}\, dt = O\left(\frac{x}{\log x}\right)$.

x "large"

$= o(x)$

$\Rightarrow \theta(x) = x + o(x)$ ✓

"(iii) $\Rightarrow$ (i)" By Abel summation:

$$\pi(x) = \sum_{n \le x} 1_P(n) \log n \cdot \frac{1}{\log n}$$

$$= \frac{\theta(x)}{\log x} + \int_{1.5}^x \frac{\theta(t)}{t (\log t)^2}\, dt$$

$$= \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) + O\left(\int_{1.5}^x \frac{1}{(\log t)^2}\, dt\right)$$

$$\int_{1.5}^{x} \frac{1}{(\log t)^2}\,dt = \int_{1.5}^{\sqrt{x}} \frac{1}{(\log t)^2}\,dt + \int_{\sqrt{x}}^{x} \frac{1}{(\log t)^2}\,dt$$

$$\ll \sqrt{x} + \frac{x}{(\log \sqrt{x})^2} \ll \frac{x}{(\log x)^2} . \qquad \blacksquare$$

Dirichlet's hyperbola method

Motivation: Estimate average for divisor function

$$\sum_{n\le x} \tau(n).$$

Apply convolution method: $\tau = \varepsilon * \varepsilon$

$$\sum_{n\le x} \tau(n) = \sum_{n \le x} \varepsilon * \varepsilon(n) = \sum_{n\le x} \sum_{d\mid n} 1$$

$$= \sum_{d\le x} \sum_{\substack{n\le x \\ n\equiv 0 (d)}} 1 = \sum_{d\le x} \left\lfloor \frac{x}{d} \right\rfloor = \sum_{d \le x} \left(\frac{x}{d} + O(1) \right)$$

$$= x \sum_{d\le x} \frac{1}{d} + O(x).$$

Recall: $\sum_{d\le x} \frac{1}{d} = \log x + \gamma + O\left(\frac{1}{x}\right)$

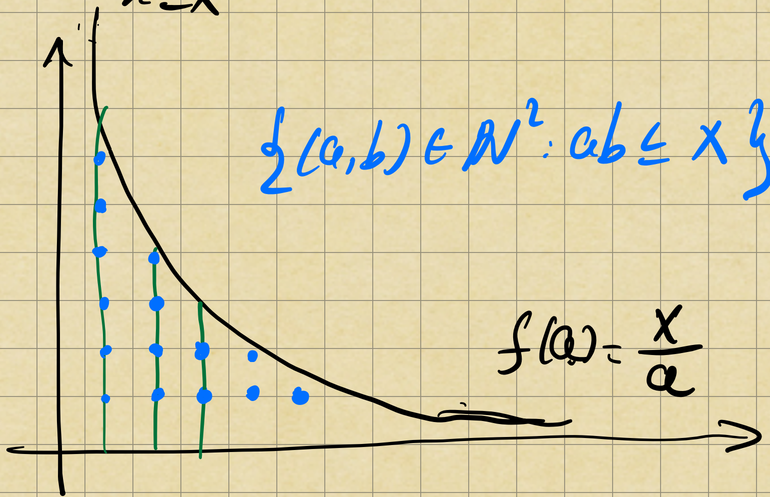
$$\Rightarrow \sum_{n \le x} \tau(n) = x \log x + O(x).$$

Can do better!

Theorem: $\sum_{n\le x} \tau(n) = x \log x + (2\gamma - 1) x + O(\sqrt{x}).$

Proof: Note that $\sum_{n \le x} \tau(n) = \sum_{n \le x} \sum_{ab=n} 1$

$$= \sum_{n \le x} |\{(a,b) \in \mathbb{N}^2 : ab = n\}| = |\{(a,b) \in \mathbb{N}^2 : ab \le x\}|.$$



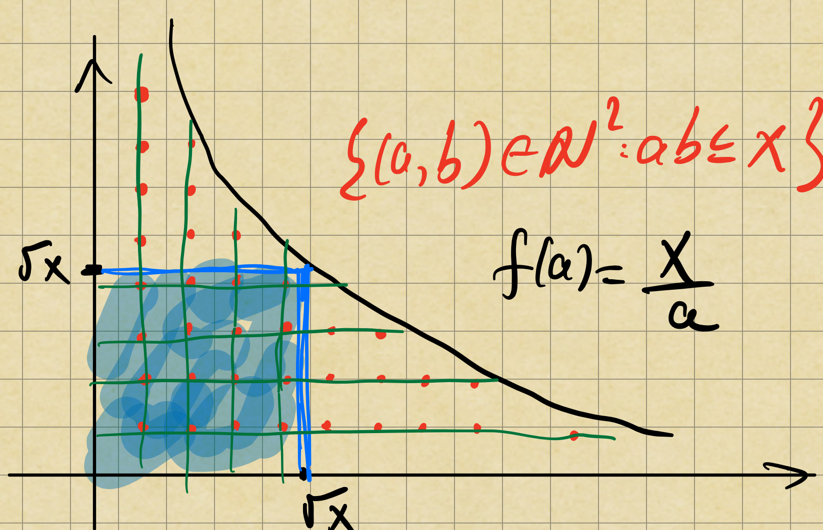
We count integer lattice points under the hyperbola!

Note: Using vertical counts

$$\sum_{n \le x} \tau(n) = \sum_{n \le x} |\{(n,b) : nb \le x\}|$$

$$= \sum_{n \le x} \lfloor \frac{x}{n} \rfloor = \sum_{n \le x} \left(\frac{x}{n} + O(1) \right) = x \lg x + O(x)$$

bad approximation, error term accumulates



Note: $ab \le x \implies a \le \sqrt{x}$ or $b \le \sqrt{x}$

Use vertical counts if $a \le \sqrt{x}$

horizontal counts if $b \le \sqrt{x}$

! Area $\{(a,b) \in \mathbb{N}^2 : a \le \sqrt{x}, b \le \sqrt{x}\}$ is double counted!

$$\sum_{n \le x} \tau(n) = \sum_{\substack{(a,b): ab \le x \\ a \le \sqrt{x}}} 1 + \sum_{\substack{(a,b): ab \le x \\ b \le \sqrt{x}}} 1 - \sum_{\substack{(a,b): ab \le x \\ a \le \sqrt{x}, b \le \sqrt{x}}} 1$$

they're equal by symmetry

$$= 2 \underbrace{\sum_{\substack{(a,b): ab \le x \\ a \le \sqrt{x}}} 1}_{(I)} - \underbrace{\sum_{\substack{(a,b): ab \le x \\ a \le \sqrt{x}, b \le \sqrt{x}}} 1}_{(II)}$$

$$(I) = \sum_{a \le \sqrt{x}} \left\lfloor \frac{x}{a} \right\rfloor = \sum_{a \le \sqrt{x}} \left(\frac{x}{a} + O(1) \right)$$

$$= x \left(\log \sqrt{x} + \gamma + O\left(\frac{1}{\sqrt{x}}\right) \right) + O(\sqrt{x})$$

$$= \frac{1}{2} x \log x + \gamma x + O(\sqrt{x})$$

$$(II) = \lfloor x \rfloor \cdot \lfloor x \rfloor = (\sqrt{x} + O(1))(\sqrt{x} + O(1))$$

$$= x + O(\sqrt{x})$$

∎

Dirichlet divisor problem: $\sum_{n\leq x} \tau(n) = x\log x + (2\gamma-1)x + \Delta(x)$.

We showed: $\Delta(x) \ll \sqrt{x}$. (Dirichlet 1849)

Classic bound: Voronoi (1904): $\Delta(x) \ll x^{1/3}\log x$

World record: Bourgain (2017): $\Delta(x) \ll_\varepsilon x^{\frac{517}{1648}+\varepsilon}$

$517/1648 = 0.3137\ldots$

Conjecture: $\Delta(x) \ll_\varepsilon x^{1/4+\varepsilon}$.